Illustration – Bootstrapping a 3 month USD Libor based IRS YC

Let there be a vanilla IRS for T years TTM and with notional of 1 USD, assume that the swap is priced to par and accordingly the MTM value of the floating leg will be equal to the MTM value of the fixed leg, there is no exchange of notional at the commencement and maturity of the IRS and here we are receiving floating and paying fixed.

MTM value of the floating leg (to follow P*r*t*DF logic):

For the float leg there will be coupon payment at time 1,2,3….k,….T; the IFRs will be L_1, L_2,…L_K,…..L_T; the door to door maturities will be d_1, d_2,….,d_K,…..d_T and the discount factors will be DF_1, DF_2,….,DF_K,…..DF_T

The MTM value of floating leg will be, $\sum_{k=1}^{k=T} 1 * L_k * d_k * DF_k$ ………. Equation (1)

⇨ $\sum_{k=1}^{k=T} L_k * d_k * DF_k$

In terms of the concept of IFRs, L_k = [$\left\{\frac{DF_{(k-1)}}{DF_k}\right\} - 1]$ / d_k

Substituting the value of L_k in Equation (1), the MTM value of floating leg will be

$$\sum_{k=1}^{k=T} [[\left\{\frac{DF_{(k-1)}}{DF_k}\right\} - 1] / d_k] * d_k * DF_k$$

⇨ $\sum_{k=1}^{k=T} \left\{\frac{DF_{(k-1)} - DF_k}{DF_k * d_k}\right\} * d_k * DF_k$

⇨ $\sum_{k=1}^{k=T} \left\{DF_{(k-1)} - DF_k\right\}$

Give that $DF_0 = 1$ and the DF terms in between will get cancelled, we end up with below representation

MTM value of floating leg = 1- DF_T ………. Equation (2)

MTM value of the fixed leg (to follow P*r*t*DF logic):

For the fixed leg there will be a standard fixed coupon payment at time 1,2,3….k,….T; the rate will be S_T; the door to door maturities will be d_1, d_2,….,d_K,…..d_T and the discount factors will be DF_1, DF_2 ,…., DF_K, …..DF_T

The MTM value of pay fixed leg will be, $-\sum_{k=1}^{k=T} 1 * S_T * d_k * DF_k$ ………. Equation (3)

(there should be a negative sign because as assumed in beginning we are paying fixed under this IRS)

Equation (3) can be further simply written as, - $S_T * \sum_{k=1}^{k=T} \ d_k * DF_k$

Given that bootstrapping is a more of a forward propagation i.e. the preceding state DF has already been worked out and substituting the sum product term $\sum_{k=1}^{k=T} \ d_k * DF_k$ by Q_T for ease of mathematical transformation (while remembering the interpretation of d_k);

The MTM value of fixed leg can be represented as - $S_T * \{Q_{(T-1)} + \ d_T * DF_T\}$ ………. Equation (4)

Remember if $DF_{(T-1)}$ is known, accordingly $Q_{(T-1)}$ is also known, now given that the IRS is at par and hence MTM of fixed leg is equal to MTM of floating let i.e. summing Equation (2) and (3) should lead to 0 (zero)

1- DF_T + [- $S_T * \{Q_{(T-1)} + \ d_T * DF_T\}$] = 0

⇨ 1- DF_T - $S_T * Q_{(T-1)} - \ S_T * d_T * DF_T\}$] = 0

⇨ 1- $S_T * Q_{(T-1)}$ = DF_T + $S_T * d_T * DF_T$

⇨ 1- $S_T * Q_{(T-1)}$ = $DF_T * (1$ + $S_T * d_T)$

⇨ DF_T = {1- $S_T * Q_{(T-1)}\}$ / (1 + $S_T * d_T$)……. Equation (5)

Equation (5) is the one that is typically used in the USD IRS curve bootstrapping for the swap segment of the IRS curve.